German solution to problem 13 "rope" (IYPT 98)

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Abstract

This is the solution of the teams Germany I and Germany II for the problem no. 13 "rope" of the 11^{th} IYPT. The problem was formulated as follows: How is it possible that a very long and strong rope can be produced from short fibers? Prepare a rope from fibers and investigate its tensile strength.

We thank Hendrik Hoeth and Markus Raichle for their theoretical help and Dr. Vogel for testing the rope.

Contents

1 Introduction

This article is divided into two main parts. The first one is only a theoretical description of the friction forces which act in the rope.

The second one deals with our self-made rope, how we made it and how we tested it. There is also an instruction how to build our model.

2 Definition of the problem

2.1 Main idea

We consider the *friction force between the fibers* as the most important force. Of course the are other forces which keep up to 40% of the strengthens of a rope. These forces are electrostatic forces and VAN-DER-WAALS-forces.

But your task was to explain how the rope can keep together, even if it consists of short fibers. For this explanation we think it's enough to consider friction forces, because it is easy to calculate them and they are already strong enough to keep the rope together. To demonstrate how strong big friction forces can be, we have made a very simple model. In this model some cords are twisted together. Half of the cords are only fixed at one side of the model, the other half at the other end so that there is no direct connection between the two ends. After twisting these cords together, they are so strong that two persons can pull at both ends. (The construction plan for our model can be found in 6.)

When solving the problem, we concentrated on some special ropes: We only used natural fibers and not artificial ones, because natural fibers are really short fibers and artificial ones can be very long. Secondly we only look on twisted ropes and not on laid or glued ones.

3 Friction force between two fibers

At first we look on two fibers which. One of them is twisted one time around the other. If we want to keep the complete length of our "rope" the second fiber must be stretched. From this stretching the normal force and the friction force between the fibers can be calculated.

3.1 Two fibers which are twisted for one time

In figure 1 the stretch of one of the fibers is shown. Of course the way around the first fiber is longer (exactly $2\pi d$) but as this makes the following formulas better understandable, we used the simplified πd instead. Another point for using this simplification is that it fits better with reality because the first fiber is of course not static when twisting a second one around.

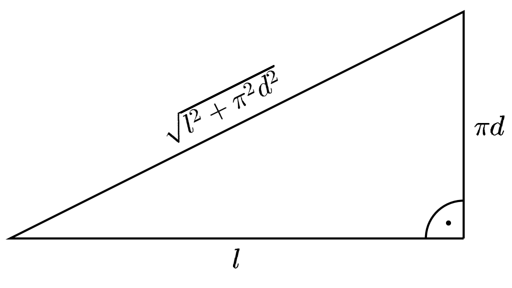

Figure 1: two fibers twisted for one time

We used the following variables or terms in our calculations:

l	:	length of the fibers
d	:	diameter of the fibers
πd	:	way around the first fiber
$\sqrt{l^2 + \pi^2 d^2}$	:	new length of the second fiber

We approximate the fibers as springs, i.e. we can use HOOK's law. The force with which we have to pull at the end of the "rope" can be calculated as follows. (Δs) is the additional length of the fiber after stretching it.)

$$F_p = D\Delta s \quad (1)$$

$$= D\left(\sqrt{l^2+\pi^2 d^2}-l\right) \quad (2)$$

Because $d \ll l$ we can make an easy approximation for this simple case:

$$F_p = Dl\left(\sqrt{1+\frac{\pi^2 d^2}{l^2}}-1\right) \quad (3)$$

$$= Dl\left(1+\frac{1}{2}\frac{\pi^2 d^2}{l^2}-1\right) \quad (4)$$

$$= \frac{1}{2}D\frac{\pi^2 d^2}{l} \quad (5)$$

3.2 Two fibers which are twisted for several times

If the two fibers are twisted several times, we don't have to change very much (see figure 2).

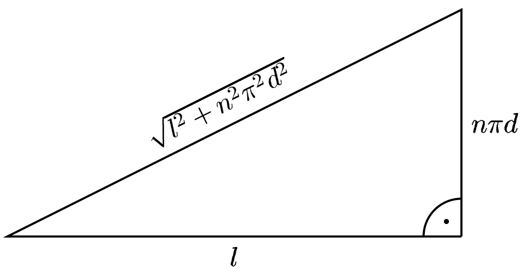

Figure 2: two fibers twisted for several times

The changes can be found in the following table:

n	:	number of twists
$n\pi d$	:	way around the first fiber
$\sqrt{l^2+n^2\pi^2 d^2}$	:	length of the second fiber

The new force can be derivate by changing formula (2):

$$F_p = D\left(\sqrt{l^2+n^2\pi^2 d^2}-l\right) \quad (6)$$

This formula should not be simplified with we approximation we made from formula (2) to (3) because n may get big, so that $\frac{\pi^2 d^2}{l^2} \ll 1$ need not to be correct.

3.3 Friction force between two fibers

If we want to know the friction force, we first have to know the normal force F_n which presses the fibers together. The calculation is based on figure 3:

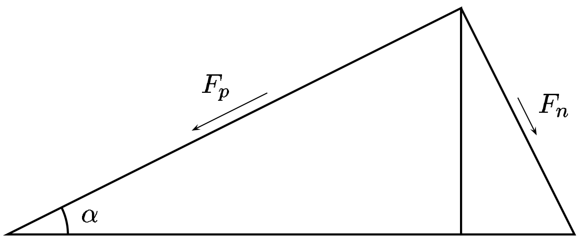

Figure 3: two fibers twisted for several times

The normal force is:

$$F_n = \tan\alpha \, F_p \tag{7}$$
$$= \frac{n\pi d}{l} F_p \tag{8}$$

The friction force therefore is:

$$F_f = f F_n \tag{9}$$
$$= f \frac{n\pi d}{l} F_p \tag{10}$$

The friction force for one twist can be derived by setting the pulling force F_p from the formula (6) into formula (10) and setting $n = 1$.

$$F_f = f \frac{\pi d}{l} \cdot \frac{1}{2} D \frac{\pi^2 d^2}{l} \tag{11}$$
$$= \frac{1}{2} f D \frac{\pi^3 d^3}{l^2} \tag{12}$$

By using formula (6) for the force we get the friction force for several twists:

$$F_f = f \frac{n\pi d}{l} D \left(\sqrt{l^2 + n^2\pi^2 d^2} - l \right) \tag{13}$$

4 Pulling force for the z$^{\text{th}}$ fiber

4.1 Derivation of the circumference

If we want to derivate the new circumference for the z$^{\text{th}}$ fiber we equal the complete cross-section area of all fibers with the new cross-section area (see figure 4). This formula is correct if the first static fiber is numbered with 0.

Sectional area of all fibers together:

$$A_f = z\pi \left(\frac{d}{2} \right)^2 \tag{14}$$

Sectional area of the rope r_r is:

$$A_o = \pi r_r^2 \tag{15}$$

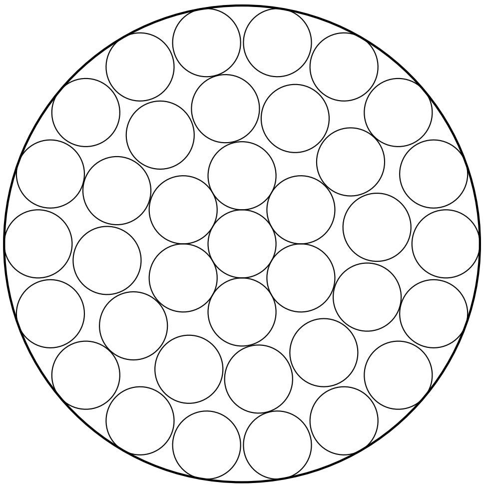

Figure 4: cross-section equation

These formulas can be equalized because the area between the fibers is nearly zero when the fibers are twisted together.

$$A_f = A_o \tag{16}$$

$$z\pi\left(\frac{d}{2}\right)^2 = \pi r_r^2 \tag{17}$$

$$\Rightarrow r_r = \sqrt{z}\,\frac{d}{2} \tag{18}$$

$$\Rightarrow d_r = \sqrt{z}\,d \tag{19}$$

$$\Rightarrow u_r = \pi d = \pi\sqrt{z}d \tag{20}$$

4.2 Force for the z^{th} fiber

With this approximation the force for the z^{th} fiber can be calculated:

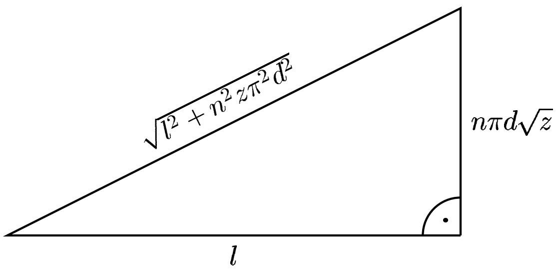

Figure 5: stretch of the z^{th} fiber

The new length is then after n twists is: $\sqrt{l^2+n^2z\pi^2d^2}$
Again the force is derivate with HOOK's law:

$$F = D\Delta s \tag{21}$$

$$= D\left(\sqrt{l^2+n^2z\pi^2d^2}-l\right) \tag{22}$$

The friction force is:

$$
\begin{aligned}
F_f &= f \tan\alpha \, F & (23) \\
&= f \frac{n\pi\sqrt{z}d}{l} D \left(\sqrt{l^2 + n^2 z \pi^2 d^2} - l \right) & (24)
\end{aligned}
$$

4.3 The complete force for several fibers

The complete friction force F_c is the sum over every F_f from formula (24):

$$
\begin{aligned}
F_c &= \sum_{z=1}^{k} \left[f \frac{n\pi\sqrt{z}d}{l} D \left(\sqrt{l^2 + n^2 z \pi^2 d^2} - l \right) \right] & (25) \\
F_c &= \frac{fDn\pi d}{l} \sum_{z=1}^{k} \left[\sqrt{z} \left(\sqrt{(l^2 + n^2 z \pi^2 d^2} - l \right) \right] & (26)
\end{aligned}
$$

When we set $c = \frac{d}{l}$, we can simplify into the following equations:

$$
\begin{aligned}
F_c &= fn\pi cD \sum_{z=1}^{k} \left[\sqrt{z} \left(\sqrt{l^2 + l^2 n^2 z \pi^2 c^2} - l \right) \right] & (27) \\
&= fn\pi cDl \sum_{z=1}^{k} \left[\sqrt{z} \left(\sqrt{1 + n^2 z \pi^2 c^2} - 1 \right) \right] & (28) \\
&= fn\pi cDl \sum_{z=1}^{k} \left[\sqrt{z} \left(\sqrt{1 + n^2 z \pi^2 c^2} - 1 \right) \right] & (29)
\end{aligned}
$$

Because $n^2 z \pi^2 c^2 \ll 1$, for $\sqrt{1 + n^2 z \pi^2 c^2}$ we can also approximated:

$$
\begin{aligned}
F_c &= fn\pi cDl \sum_{z=1}^{k} \left[\sqrt{z} \left(\frac{1}{2} n^2 z \pi^2 c^2 \right) \right] & (30) \\
&= \frac{1}{2} f n^3 \pi^3 c^3 Dl \sum_{z=1}^{k} z^{\frac{3}{2}} & (31)
\end{aligned}
$$

With $D = \frac{D_0}{l}$ this can be simplified to:

$$
F_c = \frac{1}{2} f n^3 \pi^3 c^3 D_0 \sum_{z=1}^{k} z^{\frac{3}{2}} \qquad (32)
$$

An approximation for the summation in our formula is:

$$
\begin{aligned}
\sum_{z=1}^{k} z^{\frac{3}{2}} &= \int_{0}^{k} \left(z + \frac{1}{2} \right)^{\frac{3}{2}} dz & (33) \\
&= \frac{2}{5} \left(z + \frac{1}{2} \right)^{\frac{5}{2}} & (34)
\end{aligned}
$$

As a resulting force we get:

$$\boxed{F_c = \frac{1}{5} f D_0 \, (n\pi c)^3 \left(z + \frac{1}{2}\right)^{\frac{5}{2}}} \tag{35}$$

4.4 Discussion of the formula

As this is not an easy function it should be discussed. Of course the coefficients f and D_0 have a linear influence on the force.

It is important that D_0 is a force: $D_0 = Dl$. The size of D_0 must be found experimentally or in literature.

We were not able to measure f in experiments. An estimation for it is also very complicate. Although we think useful values seem to be around 0.5, but we are not sure about that.

The number of twists n as an influence with it's 3^{rd} power. As it seems the force the rope can keep increases with the number of twists at ∞. But this is not true of course: Each fiber can only be stretched for a certain length. After this length it brakes. Therefor the force which the rope can keep gets down again, if n gets too big.

5 Our rope

5.1 Basics

We prepared our rope using electric drills and hemp. We used the electric drills instead of "professional" machines, because we wanted to prepare really an own rope. The choose for hemp as material was very easy. You can buy this kind of hemp at shops which have products for your house, because it is used under water basins.

5.2 Preparing the rope

As already said, the rope was twisted using electric drills. At the beginning, we knotted some of the hemp to it and started twisting. After a while, we added some new material to the end and continued twisting. This procedure was repeated until the rope was about the length we wanted it. Then we put a second drill to the other end and twisted in the other direction to make the rope more homogeneous.

In this way we prepared 6 laces which we twisted together to two bigger laces. These two were twisted together again.

5.3 Testing the rope

The rope was tested by Dr. Vogel at the institute "IFT – Institut für Fördertechnik" at the university of Stuttgart. The test was done with a professional rope-testing machine and lead to a force of 2.7 kN.

6 The model

6.1 Construction

For the model, you need two plates with small holes in it, some short ropes and – if you want to have a better grip – two handles.

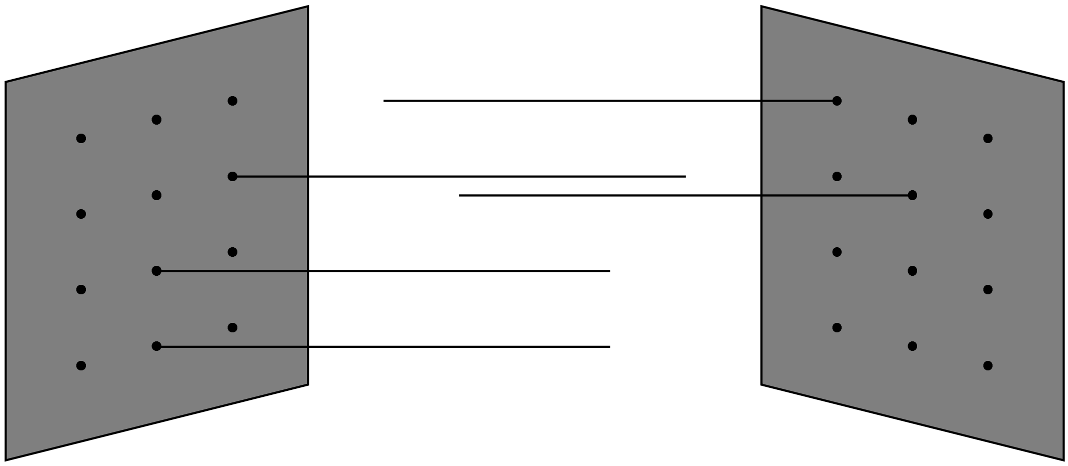

Figure 6: The model

Fix the one half of the short ropes (≈ 1.5 m) to one of the plates as symbolicly shown in figure 6, the other half to the other plate. (The figure shows it only for a few ropes fixed at one plate.)

6.2 Using the model

Lay both part now on a table so that the ropes can be mixed as shown in figure 6. Then start twisting at both ends in different directions. Mostly you'll need to keep the ropes at the beginning so that the keep together while twisting.

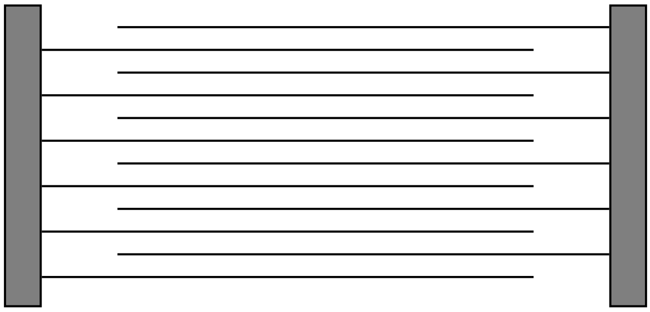

Figure 7: The model

Now you can pull very strongly at the two plates. We used handles for a better grip. Note that the forces are very strong, so all of your construction elements should be strong enough. With about 20 ropes (diameter 1 mm) we couldn't "destroy" the model with three persons.