Solution to the problem "singing glass"

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This is a summary of the report about the problem "singing glass" which was presented by the team of Germany in the final of the 12^{th} IYPT in Vienna.

Contents

1 Problem

The problem "singing glass" is the following:

> When rubbing the rim of a glass containing a liquid a tone can be heard. The same happens if the glass is immersed in a liquid. How does the pitch of the tone depend on different parameters?

This solution to it will contain the theory, some experiments and their comparisons with the theory.

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2 Theory

Simplified view:

- The glass is represented as a thin-walled cylinder, attached to a rigid circular base.
- The circumferential length of the rim is unchanged.

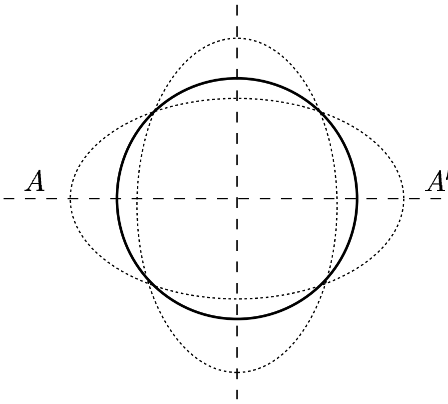

Figure 1: Basic vibration-mode

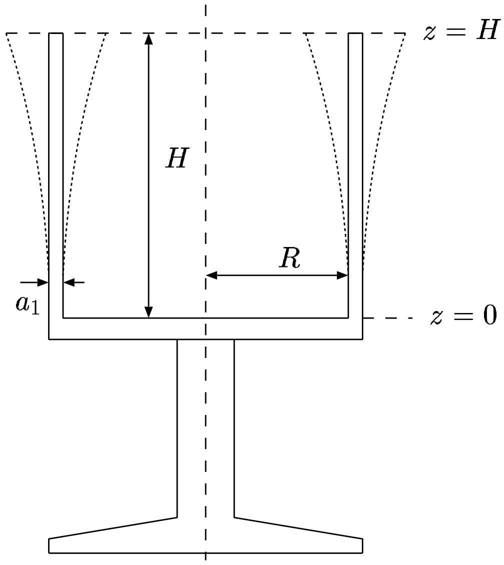

Figure 2: Vertical section

The constant rim-length forces a definit phase relationship, such that a maximum outward displacement at a particular instant is 90° away in azimuth a maximum inward displacement. For convenience let an antinodal point be on the rim at the top of the glass. It's displacement at any instant can be written

$$x(t) = x_0 \cos(\omega t) \ . \tag{1}$$

So we get for the potential energy of the vibrating glass $E_\mathrm{p} = Bx^2$ and for the total kinetic energy $E_\mathrm{k} = A\dot{x}^2$. Hence the total energy of the glass can be written as

$$E = A\dot{x}^2 + Bx^2 \ . \tag{2}$$

The requirement $E = \mathrm{const}$ leads to

$$\omega_0^2 = \frac{B}{A} \tag{3}$$

after differentiating and substituting with equation (1).
The calculation of the frequency of vibration reduces to a matter of evaluating the total kinetic and potential energies of the vibrating glass.

2.1 Kinetic energy

Lets consider the mass $\mathrm{d}m$ of an element of the wall, lying between z and $z + \mathrm{d}z$ and between ϑ and $\vartheta + \mathrm{d}\vartheta$. We get $dm = \rho_{\mathrm{g}}\ \mathrm{d}V = \rho_{\mathrm{g}} aR\ \mathrm{d}\vartheta\ \mathrm{d}z$ where ρ_{g} is the density of the glass.
For the horizontal radial displacement x of any arbitrary point of the wall of the glass can be written

$$\begin{gathered} x(z, \vartheta, t) = x_0 f(z) \cos(2\vartheta)\ \cos(\omega t) \\ 0 \leq f(z) \leq 1 \quad ; \quad 0 \leq \vartheta \leq 2\pi \end{gathered} \tag{4}$$

The masselement $\mathrm{d}m$ has then a radial velocity

$$v_r = \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega x_0 f(z) \cos(2\vartheta)\ \sin(\omega t)$$

The condition of fixed perimeter implies tangential displacement also. Because of symmetry reasons we can say that there is not tangential displacement for $\vartheta = 0$. But in general a point is displaced from the coordinates (R, ϑ) to $(R + x, \vartheta')$, where

$$\begin{aligned} \int_0^{\vartheta'} R + x(\varphi)\ \mathrm{d}\varphi &= R\vartheta \\ R\vartheta' + x_0 f(z) \cos(\omega t) \int_0^{\vartheta'} \cos(2\varphi)\ \mathrm{d}\varphi &= R\vartheta \\ R\vartheta' + \frac{1}{2} x_0 f(z) \cos(\omega t)\ \sin(2\vartheta') &= R\vartheta \end{aligned}$$

For the tangential displacement we get $s = R(\vartheta' - \vartheta)$ and so for the tangential velocity

$$v_\vartheta = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{2}\omega x_0 f(z) \sin(\omega t)\ \sin(2\vartheta)$$

Hence the kinetic energy $\mathrm{d}E_{\mathrm{k}}$ of the vibrating glass element is

$$\mathrm{d}E_{\mathrm{k}} = \frac{1}{2} v^2\ \mathrm{d}m = \frac{1}{2}\left(v_r^2 + v_\vartheta^2\right) \mathrm{d}m$$

and by integrating over ϑ we get the total kinetic energy E_{k} of the glass:

$$E_{\mathrm{k}} = \frac{5\pi}{8} \rho_{\mathrm{g}} a R x_0^2 \omega^2 \sin^2(\omega t) \int_{z=0}^{H} f^2(z)\, \mathrm{d}z \tag{5}$$

In equation (5) we see

- the major contribution to E_{k} will come from the motion of the upper parts of the wineglass.
- $x_0 \omega \sin(\omega t)$ is a velocity-expression we got by integrating over all mass-elements.

2.2 Potential energy

The calculation takes into account the energy of flexure of the glass in both horizontal and vertical planes. We consider a curved segment of material, which, when unformed, has mean radius r_0 and radial thickness a and the height b.

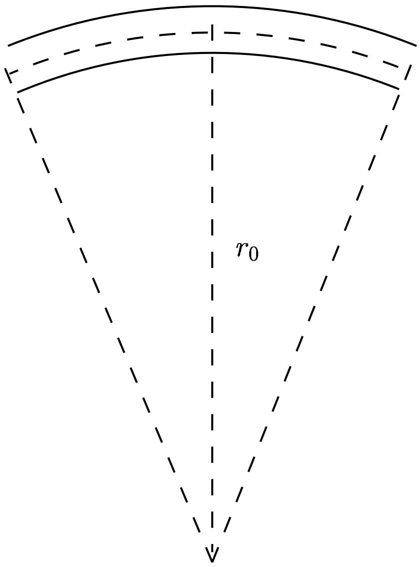

Figure 3: Stress-free glass-segment

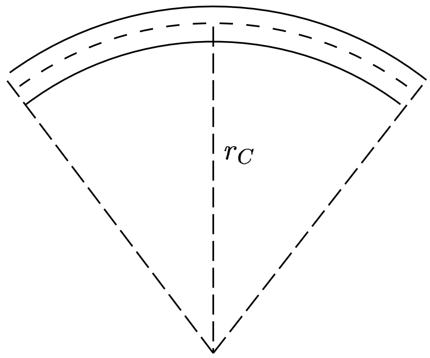

Figure 4: Stressed glass-segment

One can assume that the length ℓ_0 of the centerline of the segment remains unchanged. For a filament a distance y from this center we have a change of length

$$\delta\ell = \frac{r_0+y}{r_0}\ell_0 - \frac{r_c+y}{r_c}\ell_0 = \ell_0 y\left(\frac{1}{r_c}-\frac{1}{r_0}\right)$$

If the radial thickness of the filament is dy, and the YOUNG'S module (module of elasticity) is denoted Y, the force of tension or compression along the filament is given by

$$F = Y\,\mathrm{d}A\,\frac{\delta\ell}{\ell_0} = Y(b\ \mathrm{d}y)\frac{\delta\ell}{\ell_0} = Y(b\ \mathrm{d}y)y\left(\frac{1}{r_c}-\frac{1}{r_0}\right)$$

and the stored potential energy

$$\mathrm{d}U = \frac{1}{2}F\delta\ell = \frac{1}{2}\ell_0 bYy^2\ \mathrm{d}y\left(\frac{1}{r_c}-\frac{1}{r_0}\right)^2 \tag{6}$$

If we consider the cylinder-like glass composed by uniform bars, with maximum displacements at the top and clamped at the bottom we can take $f(z) \approx (z/H)^{3/2}$ in good agreement with our experiments. So we get

$$E_\mathrm{k} \quad = \quad \frac{5\pi}{8}\rho_\mathrm{g} aRx_0^2\omega^2\sin^2(\omega t)\frac{H}{4} \tag{7}$$

$$U = \frac{3\pi Y a^3}{8R^3} x_0^2 \cos^2(\omega t) \left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right] \frac{H}{4} \tag{8}$$

We see, that U increases significantly with increasing height H of the glass. The values for the constants A and B are

$$A = \frac{5\pi}{8} \rho_g a R \frac{H}{4}$$
$$B = \frac{3\pi Y a^3}{8R^3} \left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right] \frac{H}{4}$$

The lowest vibration-mode is given by

$$\omega_0^2 = \frac{B}{A} = \frac{3}{5} \frac{Y}{\rho_g} \frac{a^2}{R^4} \left[1 + \frac{4}{3}\left(\frac{R}{H}\right)^4\right] \tag{9}$$

The fundamental frequency $\nu_0 = \frac{\omega_0}{2\pi}$ is thus given by

$$\nu_0 = \frac{1}{2\pi} \sqrt{\frac{3Y}{5\rho_g}} \frac{a}{R^2} \sqrt{1 + \frac{4}{3}\left(\frac{R}{H}\right)^4} \tag{10}$$

We see very reasonable relations between the radius R and height H of the glass. For example the frequency decreases with increasing R.

2.3 Partially filled wineglasses

Our theory – see equation (9) – and our experiments show that the pitch of the note from a filled glass is lower than that of the same glass when empty. The added liquid is forced to participate in the vibrational motion. So the total kinetic energy of the vibrating system is increased. The potential energy remains unchanged. If we make a quantitative calculation, we must take into account, that an element of liquid, in distance r from the axis, will undergo both radial δ_r and transverse displacements δ_t. A reasonable simplification is to assume that these displacements are proportional to r:

$$\delta_r = \gamma_1 \frac{r}{R} x_0 f(z) \cos(2\vartheta) \cos(\omega t)$$
$$\delta_t = \gamma_2 \frac{r}{R} x_0 f(z) \sin(2\vartheta) \cos(\omega t)$$

γ_1 and γ_2 are formfactors. The kinetic energy of an element of the liquid of volume $r\ \mathrm{d}dr\ \mathrm{d}\vartheta\ \mathrm{d}z$ and density ρ_l can be written

$$\mathrm{d}K_l = \frac{1}{2}\rho_l (r\ \mathrm{d}r\ \mathrm{d}\vartheta\ \mathrm{d}z) v^2$$
$$\text{with}: \quad v^2 = \dot{\delta_r}^2 + \dot{\delta_t}^2$$
$$= \frac{1}{2}\rho_l (r\ \mathrm{d}r\ \mathrm{d}\vartheta\ \mathrm{d}z) \left(\frac{r}{R}\right)^2 x_0^2 f(z)^2 \omega^2 \sin^2(\omega t) \left(\gamma_1^2 \cos^2(2\vartheta) + \gamma_2^2 \sin^2(2\vartheta)\right)$$

Integrating over r, ϑ and z we get for the total kinetic energy of the liquid

$$\begin{aligned} K_\mathrm{l} &= \frac{\alpha\pi}{8}\rho_\mathrm{l}R^2x_0^2\omega^2\sin^2(\omega t)\int\limits_0^h f(z)^2\ \mathrm{d}z \qquad (11)\\ &\text{with}\quad \alpha=\gamma_1^2+\gamma_2^2 \end{aligned}$$

To this we add the kinetic energy of the glass itself. We get the total kinetic energy of the glass wenn filled up to the level $z = h$.

$$\begin{aligned} K &= K_\mathrm{l}+K_\mathrm{g}\\ &= \omega^2x_0^2\sin^2(\omega t)\left(\frac{5\pi}{8}\rho_\mathrm{g}aR\int\limits_0^H f(z)^2\ \mathrm{d}z+\frac{\alpha\pi}{8}\rho_\mathrm{l}R^2\int\limits_0^h f(z)^2\ \mathrm{d}z\right)\\ &= \omega^2x_0^2\sin^2(\omega t)\left(1+\frac{\alpha}{5}\frac{\rho_\mathrm{l}R}{\rho_\mathrm{g}a}\left(\frac{h}{H}\right)^4\right)\frac{5\pi}{8}\rho_\mathrm{g}aR\frac{H}{4} \qquad (12) \end{aligned}$$

Since the potential energy is unchanged we find that

$$\frac{\omega_0^2}{\omega_h^2}=\frac{B}{A}\cdot\frac{A'}{B}=\frac{A'}{A}\ .$$

Then the frequency ν_h of the partially filled glass is related to its frequency ν_0 when empty through the relation

$$\left(\frac{\nu_0}{\nu_h}\right)^2\approx 1+\frac{\alpha}{5}\frac{\rho_\mathrm{l}R}{\rho_\mathrm{g}a}\left(\frac{h}{H}\right)^4\ . \qquad (13)$$

Since it is more simple to determine the distance d of the liquid surface down from the top of the glass we put $h = H - d$ and take

$$\left(\frac{\nu_0}{\nu_h}\right)^2\approx 1+\frac{\alpha}{5}\frac{\rho_\mathrm{l}R}{\rho_\mathrm{g}a}\left(1-\frac{d}{H}\right)^4\ . \qquad (14)$$

3 Experiments

- **Experimental equipment**

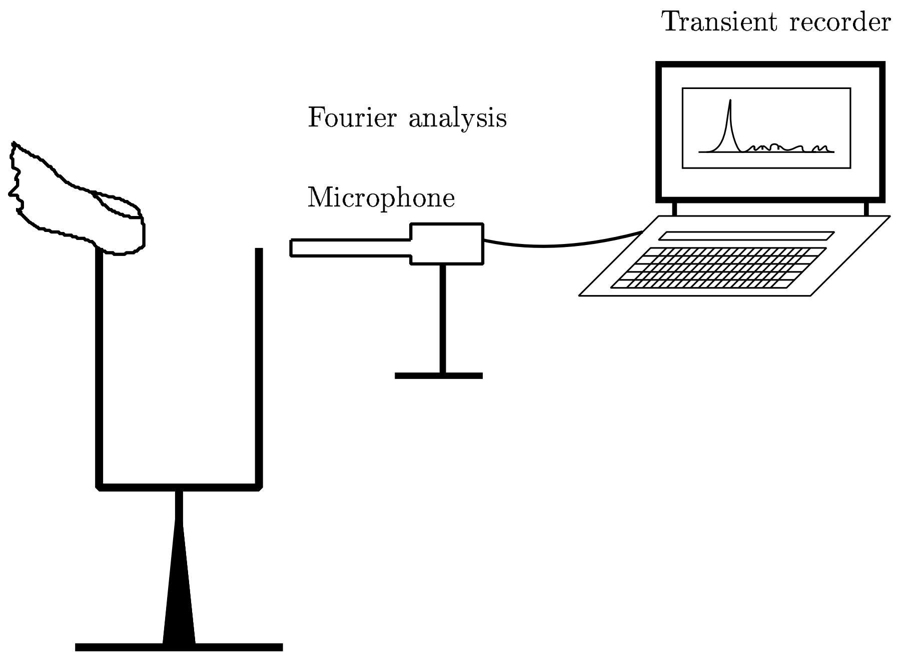

Figure 5: Experimental setup

- **Partially filles glasses**

  Shown with three transparencies in the report.

- **Glass filled with mercury**

  Shown with one transparency.

- **Glasses immersed in liquid**

  Shown with one transparency.

  The graph with the frequency spectra for the same glass

  - filled to the height h
  - immersed to the same height h

  Obviously the frequencies of the pitch agree perfectly. We already mentioned in our theory, that the elements of liquid undergo displacements only near the rim. So it was correct to make the measurements with filled glasses instead of glasses immersed in a liquid. In a good simplification, always the same mass of liquid contributes to the kinetic energy.

- **Evaluation**

  From different measurements we got for a cylindrcal wineglass the following experimental results as mean values:

$$
\begin{aligned}
R &= 0.03375\,\mathrm{m} \\
a &= 0.0015\,\mathrm{m} \\
\rho_\mathrm{l} &= 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \\
\rho_\mathrm{g} &= 3000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \\
H &= 0.0665\,\mathrm{m} \\
\nu_0 &= 1480\,\mathrm{Hz}
\end{aligned}
$$

  From these values we can evaluate the parameter $\alpha \approx 1.3361$

4 Conclusion

The experimental results show how remarkably well our theory works for a particular wineglass. Other experimental results with different glasses confirm our theory as well.